Epic math battle of history: Grothendieck vs Nikodym - Round 1

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The Grothendieck property

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Definition

A Boolean algebra $\mathbb{B}$ has the **Grothendieck property**, if $C(\mathrm{St}(\mathbb{B}))$ has the Grothendieck property.

Measures on Boolean algebras

- measure on $\mathbb{B}$ = finitely additive real-valued bounded function on $\mathbb{B}$

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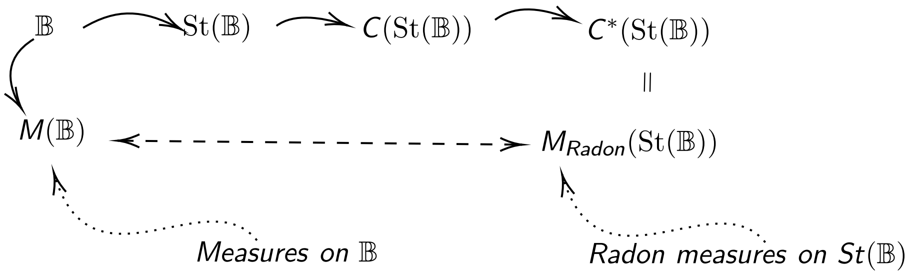

Folklore

- Every measure on $\mathbb{B}$ uniquely extends to a Radon measure on $\mathrm{St}(\mathbb{B})$
- The restriction of a Radon measure on $\mathrm{St}(\mathbb{B})$ to the clopen sets is a measure on $\mathbb{B}$

Measures on Boolean algebras

- $(\nu)_{n\in\mathbb{N}}$ on $\mathbb{B}$ is **pointwise convergent** if there exist a measure ν on $\mathbb{B}$ such that

$$\forall_{A\in\mathbb{B}}\ \nu_n(A) \to \nu(A)$$

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- **Norm** of a measure ν on $\mathbb{B}$
$$||\nu|| = |\nu|(1)$$

The Nikodym property

Definition

We say that a Boolean algebra $\mathbb{B}$ has the **Nikodym property**, if every pointwise convergent sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}$ is bounded in norm (i.e. $\sup_{n\in\mathbb{N}} \|\nu_n\| < \infty$).

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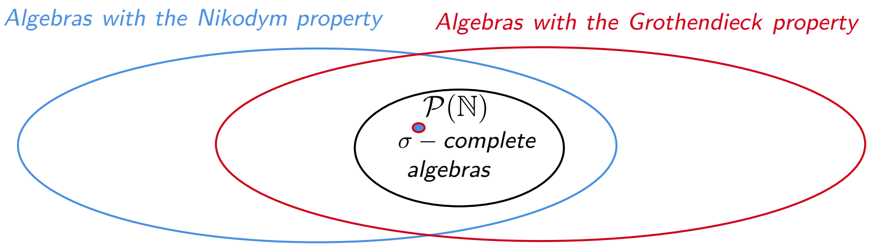
Theorem (Andô)

Complete Boolean algebras have the Nikodym property.

More examples



- σ- complete algebras have both the Nikodym and Grothendieck properties

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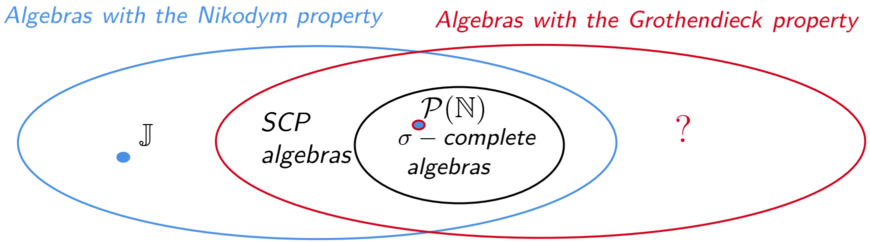

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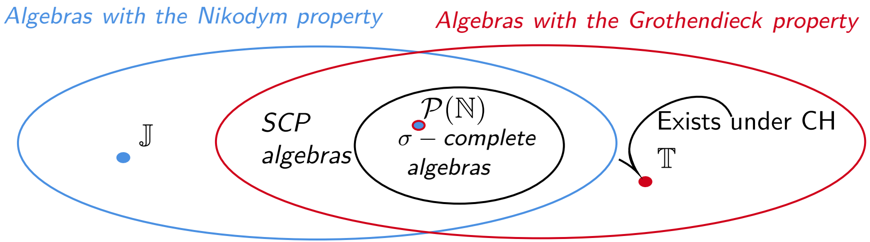

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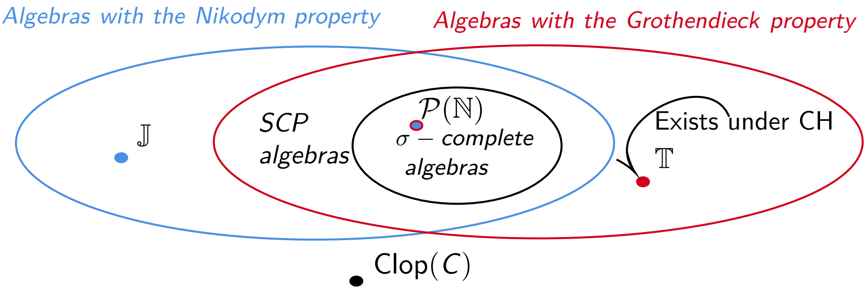

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- Talagrand (1984): Assuming CH there is a Boolean algebra $\mathbb{T}$ with the Grothendieck property and without the Nikodym property
- The algebra $\mathrm{Clop}(C)$ of all clopen subsets of the Cantor set does not have neither the Nikodym property nor the Grothendieck property

Grothendieck vs Nikodym under $\neg$CH

Open question

Is there (in ZFC) a Boolean algebra with the Grothendieck property and without the Nikodym property?

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Theorem (Głodkowski & W.)

The existence of a Boolean algebra with the Grothendieck property and without the Nikodym property is consistent with $\mathfrak{c} > \omega_1$.

Notation

$\mathrm{Clop}(C)$

The algebra $\mathrm{Clop}(C)$ of all clopen subsets of the Cantor set does not have the Nikodym property.

To show it we need some notions:

- Cantor set: $C = \{-1, 1\}^{\mathbb{N}}$

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- $\mathrm{Bor}(C)$ = the Borel subsets of C
- $\mathrm{Clop}(C)$ = the clopen subsets of C
- λ = the standard product probability measure on $\mathrm{Bor}(C)$

Notation

For $n \in \mathbb{N}$ we put $\delta_n\colon C \to \{-1, 1\}$, $\delta_n(x) = x_n$ (the n-th coordinate of x) and we define a measure φ_n on $\mathrm{Bor}(C)$ by

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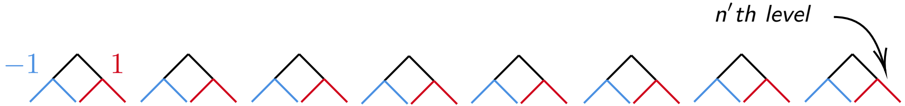

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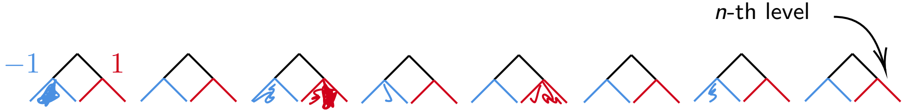

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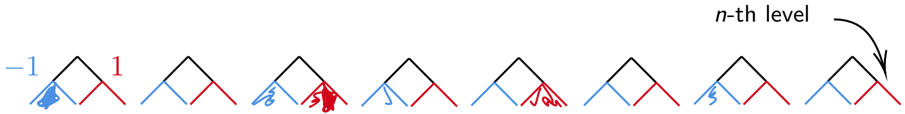

Note that for each $n \in \mathbb{N}$ we have $|\varphi_n| = \lambda$ and $||\varphi_n|| = 1$

Clopen sets

Example

$\mathrm{Clop}(C)$ does not have the Nikodym property.

A witness for the lack of the Nikodym property for $\mathrm{Clop}(C)$ is as follows:

$$\mu_n(A) = n \cdot \varphi_n(A) = n \cdot \int_A \delta_n d\lambda$$

- 🦊 (μ_n) is poinwise convergent to zero.
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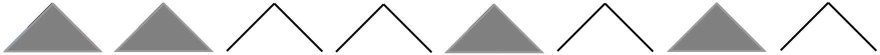

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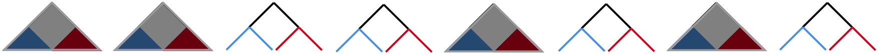

Balanced algebras

Let $m \in \mathbb{N}$ and $\varepsilon > 0$. We say that $A \in \mathrm{Bor}(C) = \mathrm{Bor}(\{-1,1\}^\omega)$ is (m,ε)-**balanced**, if for every $s \in \{-1,1\}^m$ we have

- $\frac{\lambda(A \cap \langle s \rangle)}{\lambda(\langle s \rangle)} < \frac{\varepsilon}{m}$ or $\frac{\lambda(\langle s \rangle \setminus A)}{\lambda(\langle s \rangle)} < \frac{\varepsilon}{m}$,

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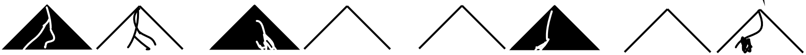

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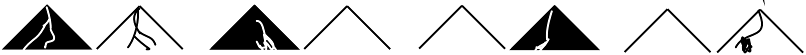

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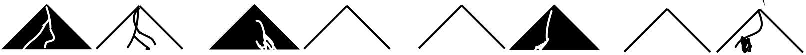

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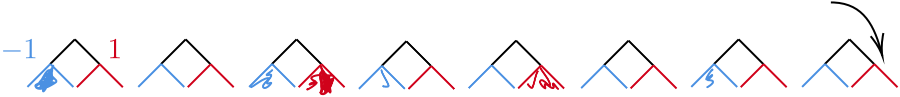

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- We say that a **Boolean algebra** $\mathbb{B} \subseteq \mathrm{Bor}(C)$ is **balanced** if for every finite family $\mathcal{A} \subseteq \mathbb{B}$ and $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that $\mathcal{A}$ is (m, ε)-balanced.

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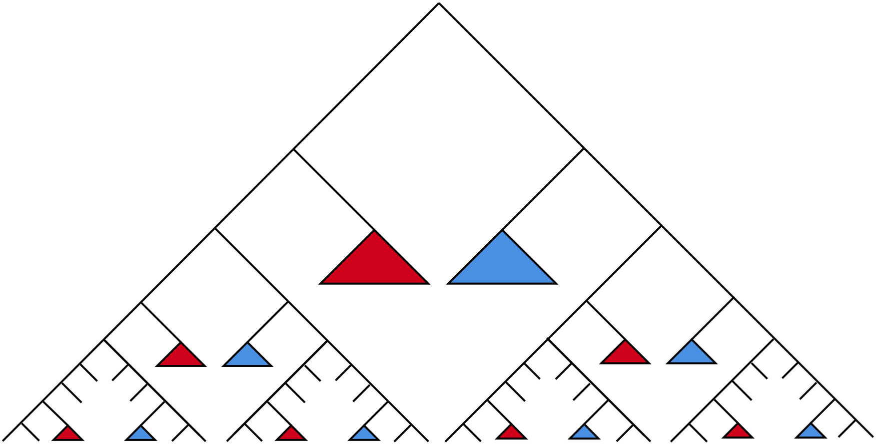
First observation

$\mathrm{Clop}(C)$ is balanced.

Examples

Second observation

There exists a balanced set which is not clopen



Example

Third observation

If $\mathbb{B} \subseteq \mathrm{Bor}(C)$ is balanced, then it does not have the Nikodym property.

To see that take $\mu_n = n\varphi_n$. The sequence $(\mu_n)_{n\in\mathbb{N}}$ is pointwise convergent to 0, but $\|\mu_n\| = n$ for every $n \in \mathbb{N}$

Extensions of countable balanced algebras

Let $\mathbb{B} \subseteq \mathrm{Bor}(C)$ be a countable balanced Boolean algebra. Suppose that

- $(\mathbb{B}_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite Boolean algebras such that

$$\bigcup_{n\in\mathbb{N}} \mathbb{B}_n = \mathbb{B}$$

- $(m_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers
- $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to 0
- $(G_n)_{n\in\mathbb{N}} \subseteq \mathbb{B}$ is a sequence of pairwise disjoint sets

and

$$\forall k \in \mathbb{N}\ \forall n \leqslant k\ \mathcal{F}\Big(\mathbb{B}_n, \bigcup_{i\leqslant k} G_i\Big) \text{ is } (m_n, \varepsilon_n)\text{-balanced}$$

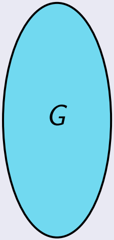
Then $\mathcal{F}(\mathbb{B}, \bigcup_{n\in\mathbb{N}} G_i)$ is balanced.

Keeping balance

Theorem (simplified version)

Let $\mathbb{B} \subseteq \mathrm{Bor}(C)$ be a balanced algebra, $m \in \mathbb{N}, \varepsilon > 0.$ Suppose that

$$G \in \mathbb{B} \text{ is } (m, \varepsilon)\text{-balanced}$$



Then there is $\theta > 0$ such that

- 🦊 for every $L \in \mathbb{B}$ such that $\lambda(L) < \theta$
- 🦊 there is a "very small" set $M \in \mathbb{B}$ such that

$$G \cup L \cup M \text{ is } (m, \varepsilon)\text{-balanced}$$

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$$L \cap M = \varnothing$$

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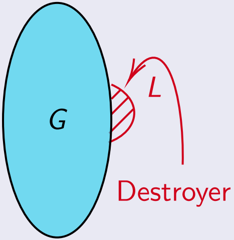

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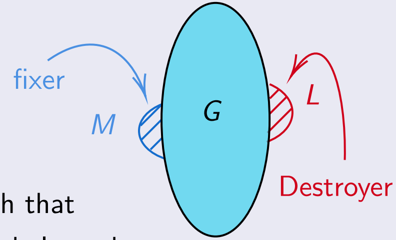

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Theorem (full version)

Let $k \in \mathbb{N}$, $\eta > 0$. Let $(m_n)_{n \leqslant k}$ be an increasing sequence of natural numbers. Let $\mathbb{B}^* \subseteq \mathbb{B} \subseteq \mathrm{Bor}(C)$ be balanced Boolean algebras and assume that $\mathrm{Clop}(C) \subseteq \mathbb{B}^*$. Let $(\mathbb{B}_n)_{n \leqslant k} \subseteq \mathbb{B}$ be finite subalgebras. Suppose that $G, P \in \mathbb{B}^*$ and the following are satisfied:

- $G \subseteq P$,
- $\forall n \leqslant k\ \mathcal{F}(\mathbb{B}_n, G)$ is $(m_n, 2^{-n})$-balanced.

Then there is $\theta > 0$ such that for every $L, Q \in \mathbb{B}^*$ satisfying

- $\max\{\lambda(L), \lambda(Q)\} < \theta$,
- $L \cap P = \varnothing$,

there is $M \in \mathbb{B}^*$ such that

- $M \cap (P \cup Q) = \varnothing$,
- $\lambda(M) < \eta$,
- $\forall n \leqslant k\ \mathcal{F}(\mathbb{B}_n, G \cup L \cup M)$ is $(m_n, 2^{-n})$-balanced.


8 hodín spánku
počas
pracovného
týždňa:
4 hodiny
spánku
cez víkend: